Elementary modes of excitation caused by the quadratic Zeeman term and the sensitivity of spin structures of small spin-2 condensates against the magnetic field

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The response of spin-2 small condensates to an external magnetic field B is studied. The parameters of the interaction are considered as variable. The emphasis is placed on clarifying the modes of excitation caused by the quadratic Zeeman term. The theoretical method used is beyond the mean field theory. A set of eigenstates with the $U(5) \supset SO(5) \supset SO(3)$ symmetry is introduced to facilitate the analysis. To obtain a quantitative evaluation on the response, the fidelity susceptibility and the B -dependent average populations of spin-components have been calculated. Mostly the particle number $N = 30$ is assumed. The effect with a larger or smaller N is also considered. It was found that the sensitivity of the response depends strongly both on the interaction and on the inherent symmetry.

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I. INTRODUCTION

The physical properties of Bose-Einstein condensates of atoms with nonzero spin can be tuned by external laser fields and magnetic fields [1]. These condensates are much less affected by defects and impurities which exist extensively in usual condensed matter systems. Therefore, the condensates can be used to simulate a variety of ideal condensed matter systems (e.g., the system of gluons) for academic studies, or to work as ideal magnetic materials in practical application (say, in quantum information and quantum computation). For the purpose of manipulation, how the condensates will react to an external magnetic field is a crucial problem to be clarified.

Usually, the particle number N in a condensate would $\geq 10^4$. If N could be greatly reduced, new physics might emerge. For an example, for spin-2 condensates, it was predicted based on the mean field theory (MFT) that the ground state would have three phases, namely, the ferromagnetic phase (f-phase), polar phase (p-phase), and cyclic phase (c-phase) [2–4]. However, from a study based on a theory beyond the MFT, the ground state may have 3, 4, or 5 phases depending on N [5]. The fine effect of N could only be seen if N is small (say, $N \leq 100$). When N is small, the properties of the condensates might depend on N sensitively, just as the properties of the nuclei depend on the number of nucleons sensitively. On the other hand, recent progress in technique suggests that the condensates with N very small could be experimentally achieved [6]. Therefore, a study of the small condensates is worthy because new knowledge additional to those from large condensates might be obtained.

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This paper is dedicated to small spin-2 condensates, and is a generalization of two previous papers [7, 8] on spin-2 and spin-1 condensates. The method adopted is beyond the MFT. The aim is to study the sensitivity of the small condensates responding to the variation of an external magnetic field B. To this aim the fidelity susceptibility [9–11] and the average population of spin-components have been calculated. To clarify the underlying physics, the set of eigenstates with the $U(5) \supset SO(5) \supset SO(3)$ symmetry firstly proposed in the refs. [5, 13] is introduced. Based on the set, elementary excitation modes caused by the quadratic Zeeman term have been found. These modes together with the associated energy gap are decisive to the response of the spin-structures against the appearance of B. It is well known that the structures of the ground states depend on the interaction. On the other hand, the ways of excitation caused by the field is found to be strongly constrained by the inherent symmetry. Thus both the interaction and symmetry together determine the response of the condensates to the field as shown below.

Since the study is not dedicated to a specific spin-2 system, the parameters of the spin-dependent interaction are considered as variable. Mostly $N = 30$ is given, the effect of a larger or smaller N is also discussed.

II. HAMILTONIAN

The atoms are assumed to be confined by an isotropic and parabolic potential $\frac{1}{2}m\omega^2 r^2$. The temperature T is assumed to be very low. Since the energy of spatial excitation of individual atom is $\sim \hbar\omega$, when $T << \hbar\omega/k_B$ it is reasonable to assume that no atoms would be spatially excited and all of them would fall into a common spatial-state $\phi(\boldsymbol{r})$. This assumption is called the single mode

approximation (SMA), which has been quite frequently used in the literatures. Furthermore, a magnetic field B lying along the Z-axis is set. When an integration is carried out over the spatial degrees of freedom, we arrive at a model Hamiltonian depending only on the spin-degrees of freedom as[5, 12]

$$H_{\rm mod} = \sum_{i<j} V_{ij} - p\sum_i \hat{f}_{iZ} + q\sum_i (\hat{f}_{iZ})^2, \quad (1)$$

where an overall constant has been neglected, $V_{ij} = \sum_S g_S \bar{n}\mathcal{P}_S^{ij}$, where S is the combined spin of the particles i and j, $\mathcal{P}_S^{ij}$ is the projector of the S-channel, g_S is the strength of interaction related directly to the s-wave scattering length of the S-channel, $\bar{n} = \int |\phi(\boldsymbol{r})|^4 \mathrm{d}\boldsymbol{r}$, and the last two terms are for the linear and quadratic Zeeman energies, respectively. Note that $\mathcal{P}_4^{ij} = 1 - \mathcal{P}_0^{ij} - \mathcal{P}_2^{ij}$. We further introduce β and θ so that $\beta\cos\theta = (g_0 - g_4)/g_4$, $\beta\sin\theta = (g_2 - g_4)/g_4$. Then we define $H'_{\rm mod} \equiv \frac{1}{\beta g_4 \bar{n}} H_{\rm mod}$. Furthermore, since M (the Z-component of the total spin) remains to be conserved under the field B, the linear Zeeman term provides only a constant and therefore can be neglected. When all the constants involved are removed, $H'_{\rm mod}$ can be rewritten as

$$H'_{\rm mod} = \sum_{i<j}(\cos\theta\ \mathcal{P}_0^{ij} + \sin\theta\ \mathcal{P}_2^{ij}) + q'\sum_i (\hat{f}_{iZ})^2, \quad (2)$$

where θ is from 0 to 2π, $q' = q/(\beta g_4 \bar{n})$. When the Hamiltonian is transformed from Eq. (1) to (2), the set of eigenenergies will be multiplied by a common constant and will be further shifted as a whole. But the eigenstates will remain unchanged. Since only two parameters θ and q' are contained in $H'_{\rm mod}$, related analysis is easier to perform.

When $q' = 0$, it has been proved that $H'_{\rm mod}$ can be rewritten as a sum of four Casimir operators of a chain of groups $U(5) \supset SO(5) \supset SO(3)$ [5]. Therefore the problem can be solved analytically based on the group algebra [5, 13, 14]. Three quantum numbers, namely, F (total spin), M (its Z-component), and v (seniority, $N-v$ must be an even integer, and $(N-v)/2$ is the number of singlet pairs) are introduced to classify the states. The eigenenergies related to these quantum numbers are

$$E_{vF} = \frac{10\sin\theta - 7\cos\theta}{70} v(v+3) - \frac{\sin\theta}{14} F(F+1), \quad (3)$$

where an irrelevant overall constant depending on N has been further neglected. The associated eigenstates are denoted as Ψ_{vFM}. Note that, for specifying a symmetry-adapted eigenstate, not all the (v, F) pairs are allowed. Some of them are prohibited by inherent symmetry. The details are referred to [13]. Eq. (3) leads to a notable feature, namely, when $\sin\theta = \frac{7}{10}\cos\theta$, all the symmetry-adapted Ψ_{vFM} with the same F are degenerate. Whereas when $\sin\theta = 0$, all the symmetry-adapted Ψ_{vFM} with the same v are degenerate. If the group of degenerate states happens to be the ground states, the low-temperature behavior of the condensate would be seriously affected.

In general, a slight increase of N (say from being even to odd) will affect the structure of the ground state . However, when N is large, the effect is weak. For the convenience of discussion, it is assumed that N is a multiple of 2 and 3 in the follows (other cases of N can be similarly discussed). When θ and M are fixed ($M \geq 0$ is assumed), from Eq. (3) and from the symmetry-adaptability, the pair (v_g, F_g) for the ground state (the lowest state with the given M) can be determined. When $M = 0$, based on the method of group algebra, there are three regions of θ [5]. When θ is from 0 to $\theta_{fp} \equiv \arctan[-\frac{7(N+3)}{10(N-2)}] < \pi$ (region I), $v_g = N$ and $F_g = 2N$, and accordingly the ground state is in the f-phase. In this phase all the spins are nearly aligned along a common direction to form a total spin, which is vertical to the Z-axis (if $M = 0$) and has an arbitrary azimuthal angle. When θ is from θ_{fp} to $\theta_{pc} \equiv \arctan(\frac{7}{10}) = 214.99°$ (region II), $v_g = 0$ and $F_g = 0$, and accordingly in the p-phase, in which the spins are two-by-two to form the singlet pairs (If N is odd, the ground state would have $v_g = 1$ and $F_g = 2$, and accordingly would be composed of a group of singlet pairs together with a single particle. This case is not discussed in this paper). When θ is from θ_{pc} to 2π (region III), $v_g = N$ and $F_g = 0$, and accordingly in the c-phase, in which the spins are essentially three-by-three to form the triplexes (If N is not a multiple of 3, additional particle(s) would be added)[5]. The division into three regions based on the group algebra coincides with that from the MFT when N is sufficiently large[2, 4]. Nonetheless, the details of spin correlations can be understood from the former but can not from the MFT.

When $q' \neq 0$, there is no analytical solution. F is no more conserved, but M remains to be a good quantum number. To obtain numerical solution, the Fock-states $|\alpha\rangle = |N_2^\alpha, N_1^\alpha, N_0^\alpha, N_{-1}^\alpha, N_{-2}^\alpha\rangle$ are used as basis functions for the diagonalization of $H'_{\rm mod}$, where N_μ^α is the number of particles in the μ-spin-component, $\sum_\mu N_\mu^\alpha = N$ and $\sum_\mu \mu N_\mu^\alpha = M$, $|\alpha\rangle$ as a whole form a complete set for symmetrized spin-states. Thereby exact eigenstates $\psi_{iM}^{q'}$ of $H'_{\rm mod}$ can be obtained, where i is a serial number of the series of states with the given M ($i = 1$ denotes the ground state). Various information will be extracted from $\psi_{iM}^{q'}$. The case with $M = 0$ is firstly considered.

III. THE FIDELITY SUSCEPTIBILITY OF THE GROUND STATE AND THE MODE OF EXCITATION

Firstly, we would like to study the sensitivity of the ground state $\psi_{1M}^{q'}$ against the variation of the field. For this purpose the fidelity susceptibility [9–11]

$$\chi_M(q') = \lim_{\varepsilon\to 0}\frac{2}{\varepsilon^2}(1 - |\langle\psi_{1M}^{q'+\varepsilon}|\psi_{1M}^{q'}\rangle|), \quad (4)$$

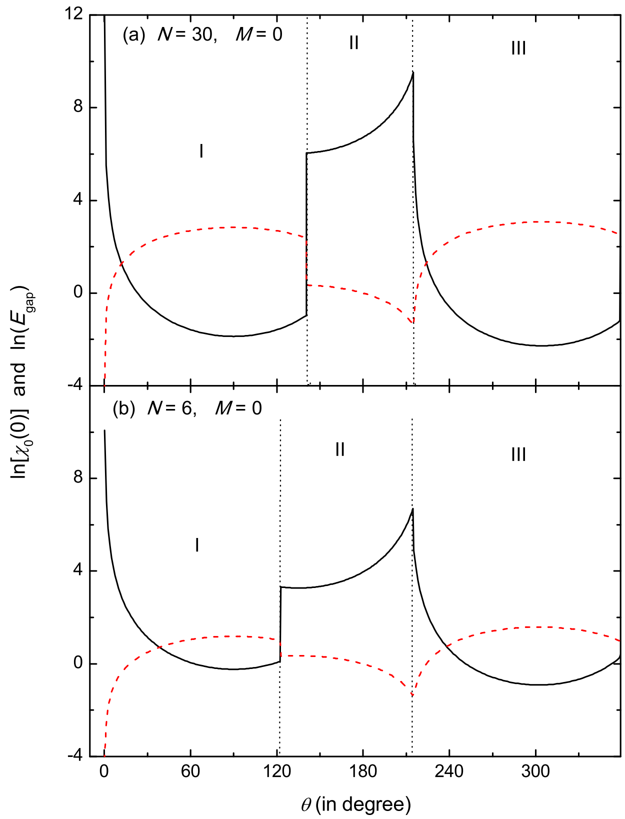

FIG. 1: (Color online.) $\ln[\chi_0(0)]$ (solid) and $\ln(E_{\text{gap}})$ (dash) against θ. $M=0$ is assumed. The two vertical dotted lines located at θ_{fp} and θ_{pc}, respectively, are introduced for separating the regions I, II, and III. In (a) $N=30$ and $\theta_{fp}=140.48°$, while in (b) $N=6$ and $\theta_{fp}=122.41°$. In both (a) and (b), $\theta_{pc}=214.99°$.

has been calculated. A larger $\chi_M(q')$ implies that the deviation between $\psi_{1M}^{q'+\varepsilon}$ and $\psi_{1M}^{q'}$ is larger, thus the state responds more sensitively to q'. Therefore this quantity measures quantitatively the sensitivity. From the eigenstates $\psi_{1,0}^{\varepsilon}$ and $\psi_{1,0}^{0}$ (both can be obtained via the diagonalization), $\ln[\chi_0(0)]$ of the ground state as a function of θ is shown by the solid curves in Fig. 1. The behaviors of $\chi_0(0)$ in the three regions of θ, which are separated by the dotted vertical lines, are different. In general, the p-phase in region II is much more sensitive to the appearance of the magnetic field than the f- and c-phases. From Eq. (3), we know that the ground state is degenerate when $\theta=\theta_{pc}$ or 0. In these cases, the susceptibility can not be well defined. Therefore, in Fig. 1, the two narrow domains $(\theta_{pc}-\delta,\theta_{pc}+\delta)$ and $(-\delta,\delta)$ with $\delta=\pi/180$ are actually not included in the calculation. Nonetheless, when θ is close to these narrow domains, the sensitivity is very high.

In order to understand the behavior shown in Fig. 1, we have to study the mode of excitation caused by the quadratic Zeeman term. The rule of selection governing the matrix elements

$$Q_{v'F',vF}^{M}=\langle\Psi_{v'F'M}|\sum_i(\hat{f}_{iZ})^2|\Psi_{vFM}\rangle, \tag{5}$$

is crucial. One can prove that $Q_{v'F',vF}^{M}$ is nonzero only if $v'-v=0$ or ± 2 and $|F'-F|\le 2$. Additionally, if $M=0$, $(-1)^{F'}=(-1)^F$ is required. Of course, both the pairs (v',F') and (v,F) should be symmetry-adapted. Let the excitation caused by $Q_{v'F',vF}^{M}$ be called the elementary excitation. For the ground state $\Psi_{v_gF_g0}$, due to the symmetry constraint[15], one can prove that the non-diagonal $Q_{v'F',v_gF_g}^{0}$ is nonzero only if $(v',F')=(\bar{v}_g,\bar{F}_g)$, where $(\bar{v}_g,\bar{F}_g)=(N,2N-2)$, $(2,2)$, and $(N-2,2)$, respectively, for the f-, p-, and c-phases (refer to Tab. I). It implies that each phase has only a unique mode of elementary excitation. This is a distinguished feature of $M=0$ states.

When the ground state is not close to being degenerate, the first-order perturbation theory can be used to calculate the susceptibility. When $q'=0$ and ε is very small, due to having only a unique elementary mode, the first order approximation of $\psi_{1,0}^{\varepsilon}$ can be written as

$$\psi_{1,0}^{\varepsilon}=\gamma(\Psi_{v_gF_g,0}+\varepsilon\frac{Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}}{E_{\text{gap}}}\Psi_{\bar{v}_g\bar{F}_g,0}), \tag{6}$$

where $E_{\text{gap}}=E_{\bar{v}_g\bar{F}_g}-E_{v_gF_g}$, γ is simply a constant for the normalization and can be easily obtained. Inserting Eq. (6) into (4), we have

$$\chi_0(0)\approx(\frac{Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}}{E_{\text{gap}}})^2. \tag{7}$$

From Eq. (3), we know that, for the f-phase $E_{\text{gap}}=E_{N,2N-2}-E_{N,2N}=\frac{4N-1}{7}\sin\theta$, for the p-phase $E_{\text{gap}}=E_{2,2}-E_{0,0}=\sin\theta-\cos\theta$, and for the c-phase $E_{\text{gap}}=E_{N-2,2}-E_{N,0}=\frac{7\cos\theta-10\sin\theta}{70}(4N+2)-\frac{3\sin\theta}{7}$. E_{gap} are also plotted in Fig. 1 by the dash curves. In general, Eq. (7) gives a very good approximation except that θ is close to θ_{pc} or 0 (2π). When θ varies within a region, $Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}$ does not depend on θ. Therefore the θ-dependence is due to the factor $(1/E_{\text{gap}})^2$ and can be shown by comparing the solid and dash curves. Incidentally, since $\Psi_{\bar{v}_g\bar{F}_g,0}$ might not be the first excited state, the gap defined here might not be the energy difference between the ground and the first excited states. Furthermore, when q' is not very weak, one has to go beyond the first order perturbation theory, and therefore, in addition to the elementary excitation, higher order excitations will be included. Accordingly, various $\Psi_{v',F',0}$ components will be contained in $\psi_{1,0}^{q'}$.

The features of the ground states with $M=0$ are summarized in Tab. I. Some details in the table may be changed if N is not a multiple of 2 and 3 (Say, if N is odd, θ_{fp} is changed to $\arctan[\frac{-7(N+4)}{10(N-1)}]<\pi$).

Note that the elementary excitation of the f-phase in region I, namely, from $(v_g=N,F_g=2N)$ to $(N,2N-2)$,

TABLE I: The three phases of the ground states with $M = 0$ and the associated three connected regions of θ. N is assumed to be a multiple of 2 and 3. (v_g, F_g) are the quantum numbers of the ground state, $(\bar{v}_g, \bar{F}_g)$ specifies the corresponding unique mode of elementary excitation, and E_{gap} is the associated energy of excitation. $Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}$ is the matrix element of the elementary excitation caused by the quadratic Zeeman term $\sum_i(\hat{f}_{iZ})^2$. This matrix element is given at $N = 30$ and 6 (inside the parentheses).

region	I (ferromagnetic)	II (polar)	III (cyclic)
right border	$\theta_{fp} = \arctan[\frac{-7(N+3)}{10(N-2)}] < \pi$	$\theta_{pc} = \arctan(\frac{7}{10}) > \pi$	$\theta_{cf} = 2\pi$
v_g, F_g	$N, 2N$	$0, 0$	$N, 0$
$\bar{v}_g, \bar{F}_g$	$N, 2N-2$	$2, 2$	$N-2, 2$
E_{gap}	$\frac{4N-1}{7}\sin\theta$	$\sin\theta - \cos\theta$	$\frac{7\cos\theta-10\sin\theta}{70}(4N+2) - \frac{3\sin\theta}{7}$
$Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}$	6.68 (2.93)	28.98 (7.27)	6.93 (3.10)

is essentially a change of the total spin. The associated energy gap is nearly proportional to N. Due to the enlargement of the gap by the factor N, the f-phase has a very low susceptibility in general. However, when $\theta \to 0$, the gap tends to zero resulting in a great increase in the susceptibility. This explains the sharp peak appearing in the left end of Fig. 1a or 1b. Nonetheless, since the gap will increase with N, the sharp peak will be more and more suppressed when N is larger and larger.

The elementary excitation of the p-phase in region II, namely, from $(0, 0)$ to (2,2), is a transformation of a singlet pair into a $S = 2$ pair. The associated $Q^0_{22,00}$ is roughly proportional to N [16]. Even when N is not large, it is still much larger than the $Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}$ of I and III as shown in Tab. I. Furthermore, the gap is small and does not depend on N. Therefore, the p-phase has a very high susceptibility, and will become higher when N is larger. The minimum of the gap appears at the right border of II. Accordingly, there is a peak [17].

The elementary excitation of the c-phase in III, namely, from $(N, 0)$ to $(N-2, 2)$ is essentially a transformation of a triplex to a singlet pair plus an extra particle. The associated gap is also enlarged by N. It will be in general large, therefore the c-phase has also a very low susceptibility. However, the gap would decrease very fast if θ is close to θ_{pc}. Accordingly, there is a sharp peak at the left border of III as shown in Fig. 1. Up to now the feature in Fig. 1 has been explained.

Comparing Fig. 1a with 1b we know that the qualitative feature of the susceptibility does not change with N. However, since θ_{fp} depends on N, the border between I and II will move a little to the right when N becomes larger and will tend to $\arctan(-7/10) = 145.01°$ when $N \to \infty$ (this value coincides with that from the MFT). Furthermore, the increase of N will cause a remarkable increase of $\chi_0(0)$ in II (due to the fact that $Q^0_{2,2,0,0}$ is roughly proportional to N) and will cause a small decrease of $\chi_0(0)$ in I and III (Although $Q^0_{\bar{v}_g\bar{F}_g,v_gF_g}$ increases with N, however the gap increase also. The two effects compete with each other, and finally the latter over takes the former).

IV. POPULATIONS OF SPIN COMPONENTS UNDER A MAGNETIC FIELD

When a magnetic field is applied, since the quadratic Zeeman term imposes additional energy to the particles with $\mu \neq 0$, the ground state will prefer to have the number of $\mu = 0$ particles, N_0, larger so as to reduce the energy. Thus, in addition to the susceptibility, the study of the variation of N_0 is also a way to understand the effect of the field. Note that the experimental measurement of the susceptibility might be difficult. On the contrary, N_0 can be easily measured, say, via the Stern-Gerlach technique.

Let the eigenstates of H'_{mod} be expanded in terms of the Fock-states as $\psi^{q'}_{iM} = \sum_\alpha c^{q'iM}_\alpha|\alpha\rangle$. One can extract a particle (say, particle 1) from a Fock-state as

$$|\alpha\rangle = \sum_\nu \eta_\nu(1)\sqrt{\frac{N^\alpha_\nu}{N}}|\cdots, N^\alpha_\nu - 1, \cdots\rangle, \tag{8}$$

where η_ν is the spin-state of a particle in the component ν, $|\cdots, N^\alpha_\nu - 1, \cdots\rangle$ is a Fock-state of the $(N-1)$-body system, in which the number of particles in the component ν decreases by one. Therefore, the particle can also be extracted from the eigenstate as

$$\psi^{q'}_{iM} \equiv \sum_\nu \eta_\nu(1)\psi^{q'iM}_\nu, \tag{9}$$

$$\psi^{q'iM}_\nu = \sum_\alpha c^{q'iM}_\alpha\sqrt{\frac{N^\alpha_\nu}{N}}|\cdots, N^\alpha_\nu - 1, \cdots\rangle. \tag{10}$$

Thereby the probability of a particle in ν is just

$$P^{i,M}_\nu \equiv \langle\psi^{q'iM}_\nu|\psi^{q'iM}_\nu\rangle. \tag{11}$$

They fulfill $\sum_\nu P^{i,M}_\nu = 1$. $P^{i,M}_\nu$ is called the 1-body probability, and $NP^{i,M}_\nu \equiv \bar{N}_\nu$ is just the average population of the ν component of the i-th state.

The one-body probabilities $P^{1,0}_0$ of the ground states with $M = 0$ and q' given at a number of values are plotted in Fig. 2. When $q' = 0$ (solid curve), the ground state

has three choices of phases as mentioned. The eigen-spin-state of the f-phase is

$$\psi_{1,0}^{q'=0} = \Psi_{N,2N,0} = (\{[(\eta\eta)_4\eta]_6\eta\}_8\cdots)_{2N,0}. \qquad (12)$$

where, the special way of spin coupling (i.e., the combined spin of an arbitrary group of j particles is $2j$) assures that the spin-state written in Eq. (12) is normalized and symmetrized. It is straight forward to extract a particle from the form of Eq. (12), thereby the 1-body probability of the f-phase can be obtained as $P_\nu^{1,0} = (C_{2,\nu;\ 2N-2,-\nu}^{2N,0})^2$, where the Clebsch-Gordan coefficient has been introduced. When $\nu = 0$, $P_0^{1,0} = \frac{6N(2N-1)^2}{(4N-1)(4N-2)(4N-3)}$. This value is equal to 0.381 if $N = 30$ as shown in Fig. 2, and will tend to $3/8 = 0.375$ if $N \to \infty$.

Both the p- and c-phases have $F = 0$ (if $M = 0$), and therefore no special orientation is preferred. Accordingly, as shown in [7], $P_\nu^{1,0} = 1/5$ for all ν disregarding how N is. When the field is applied, both the f- and c-phases are inert to the increase of q', while the p-phase is extremely sensitive. These coincide with the previous findings in the susceptibility. In particular, when θ is close to the right border of II, a great increase of $\bar{N}_0$ appears even if the field is still weak. In I and III the increase of $\bar{N}_0$ is very slight except when θ is close to 0 (2π) and θ_{pc}, where the ground state is nearly degenerate as mentioned. The great difference in sensitivity with respect to q' holds disregarding how N is. Since the interactions of some realistic spin-2 atoms (say, ^{87}Rb, ^{23}Na, and ^{85}Rb [2]) might have $\theta \approx \theta_{pc}$, the high sensitivity in this particular domain would be very helpful to the exploration of the interactions.

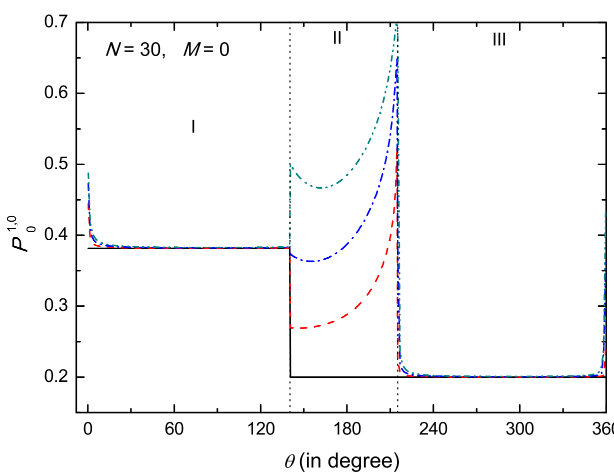

FIG. 2: (Color online.) $P_0^{1,0}$ against θ with $N = 30$ and $M = 0$. The solid, dash, dash-dot and dash-dot-dot curves have $q' = 0$, 0.01, 0.02, and 0.03, respectively.

Incidentally, if N increases from 30 to ∞, the general feature of Fig. 2 would remain unchanged. However, the border between the regions I and II would shift from 140.48° to 145.71°, the solid line in I (for $q' = 0$) would be lower a little from 0.381 to 0.375, the parabolic-like curves in II for the p-phase would lie remarkably higher (say, for $M = 0$, $\theta = \pi$, and $q' = 0.03$, $P_0^{1,0}$ would be 0.282, 0.485, and 0.651 if $N = 6$, 30, and 60, respectively).

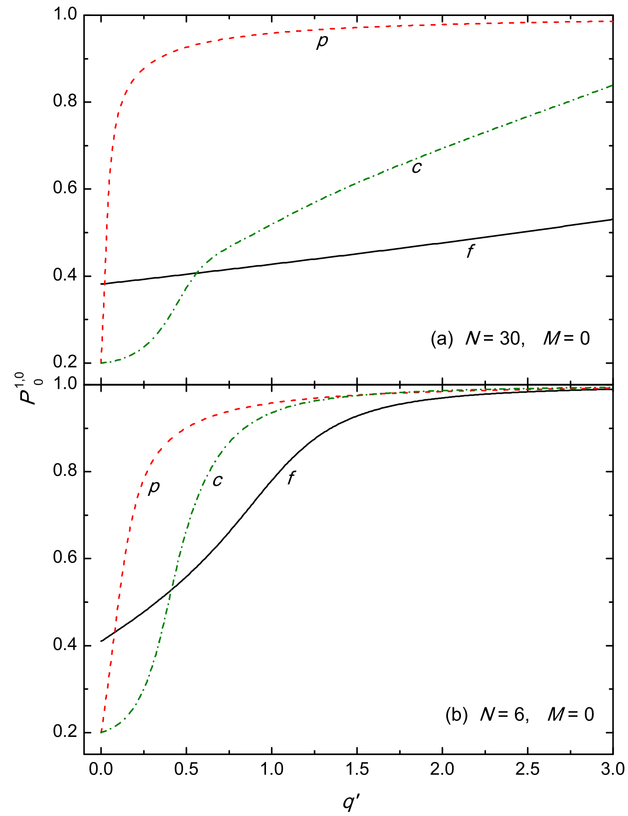

FIG. 3: (Color online.) $P_0^{1,0}$ against q' with $N = 30$ (a) and 6 (b), and $M = 0$. The solid, dash, and dash-dot curves have $\theta = \pi/2$, π, and $3\pi/2$, and represent the ferromagnetic (f), polar (p), and cyclic (c) phases, respectively.

It was found that, when $q' \to \infty$, $\lim\langle 0, 0, N, 0, 0|\psi_{10}^{q'}\rangle^2 = 1$ disregarding how θ is. It implies that all the ground states will tend to the same Fock-state $|0, 0, N, 0, 0\rangle$ as the strong field limit, where all the spins have $\mu = 0$. Accordingly, $P_\mu^{1,0} \to 1$ (if $\mu = 0$) or $\to 0$ (if $\mu \neq 0$). The process of going to the limit is shown in Fig. 3. Where θ is given at three values $\pi/2$, π, and $3\pi/2$ to represent the f-, p-, and c-phases, respectively. One can see that the dash curve for the p-phase has a very steep take off at the left end. It implies that the p-phase needs only a relatively much weaker field to push it to its limit. This coincides with the finding from Fig. 2. When N is larger, the take off is steeper (e.g., when q' increases from 0 to 0.25, $P_0^{1,0}$ would increase from 0.200 to 0.783, 0.877, and 0.905, respectively, if $N = 6$, 30, and 60). It implies that, the p-phase of a larger condensate would be more sensitive to the field as found before.

On the other hand, the take off of the dash-dot curve (c-phase) is much milder, and the solid curve (f-phase) is

the mildest, and they would be even milder if N is larger. For examples, when $q' = 0.25$, the dash-dot curves would be 0.300, 0.237, and 0.230, and the solid curves would be 0.478, 0.393, and 0.384, if $N = 6$, 30, and 60, respectively. It implies that, the f- and c-phases of a larger condensate would be more inert to the field than a smaller one.

V. THE CASE WITH NONZERO MAGNETIZATION

In the previous sections essentially the case with $M = 0$ is considered. When $q' = 0$ and $M > 0$, based on Eq. (3) together with the constraints $2v \geq F \geq M$ and $(-1)^N = (-1)^v$, the dependence of the quantum numbers of the ground state (v_g, F_g) on θ are listed in Tab. II. The effect of M is embodied by an integer v_0, which is the smallest even (odd) integer $\geq M/2$ if N is even (odd), as shown in the table. Instead of three, there are four phases. However, when $M = 4k$ $(k = 0, 1, 2, \cdots)$ and N is even, or $M = 4k + 2$ and N is odd, we have $2v_0 = M$. In this case, the two phases polar-A and -B are combined, and the four phases reduce to three as before.

Note that, when M is ≤ 5, (v_0, M) and (N, M) might be prohibited by symmetry [15]. Once (v_0, M) or (N, M) is prohibited, other more advantageous symmetry-adapted (v, F) pair will replace it (Say, in $\mathrm{II_B}$ where a smaller v and a smaller F will lead to a lower energy, if (v_0, M) is prohibited, it would be replaced by $(v_0, M + 1)$ and/or $(v_0 + 2, M)$. In III where a larger v and a smaller F will lead to a lower energy, if (N, M) is prohibited, it would be replaced by $(N - 2, M)$ and/or $(N, M + 1)$). In these cases the appearance of more than four phases is possible.

Just for an example, we choose N =even and $M = 7$ (this choice is rather arbitrary). Then, we have $v_0 = 4$. Note that, for the region $\mathrm{II_B}$, $(v_g, F_g) = (4, 7)$ is prohibited. Therefore, it would be replaced by $(4, 8)$ and/or $(6, 7)$. Note that, from Eq. (3), we would have $E_{4,8} = E_{6,7}$ if $\theta = \arctan(\frac{91}{170}) = 208.16°$. Accordingly, the $\mathrm{II_B}$ splits into two. When $\pi \leq \theta < \arctan(\frac{91}{170})$, the symmetry-adapted ground state will have $v_g = 4, F_g = 8$. Thus this part becomes an extension of $\mathrm{II_A}$, and therefore the right border of $\mathrm{II_A}$ extends from π to $\arctan(\frac{91}{170})$. When $\arctan(\frac{91}{170}) < \theta \leq \arctan(\frac{7}{10}) = 214.99°$, $v_g = 6, F_g = 7$. Thus the region $\mathrm{II_B}$ with the quantum numbers $(6, 7)$ becomes very narrow.

The fidelity susceptibility against θ is shown in Fig. 4, where the division into four regions is clear (including the very narrow region $\mathrm{II_B}$). In region I for the f-phase, the eigenstate $\Psi_{N,2N,M} = (\{[(\eta\eta)_4\eta]_6\eta\}_8\cdots)_{2N,M}$, where the spins are nearly lying along a common direction to form a total spin, which has an arbitrary azimuthal angle but not lying in the X-Y plane. In $\mathrm{II_A}$ and $\mathrm{II_B}$, the ground states are still dominated by the singlet pairs together with v_0 unpaired particles. In $\mathrm{II_A}$, the subsystem of the unpaired particles is in the f-phase, i.e., they form a spin-state as $(\{[(\eta\eta)_4\eta]_6\eta\}_8\cdots)_{2v_0,M}$. In $\mathrm{II_B}$, the

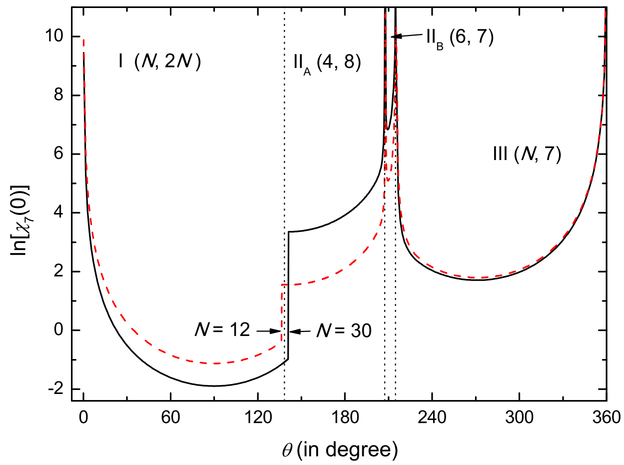

FIG. 4: (Color online.) $\ln(\chi_7(0))$ against θ with $N = 30$ (solid) and 12 (dash). $M = 7$ is assumed. The domain of θ are divided into four regions separated by the dotted vertical lines (The dotted line at the left is simply the average of the two cases with $N = 30$ and 12). The quantum numbers of the ground states (v_g, F_g) in each region are marked.

spins of the unpaired particles are slightly diffused from a common direction, because their total spin M is slightly smaller than $2v_0$. In III, the c-phase is dominated by the triplexes together with a few unpaired particles. One can see that the susceptibilities of the pA- and pB-phases are in general much higher, and would become even higher if N is larger. This is similar to the findings from Fig. 1. In particular, when θ is close to the borders of the regions (except the border between I and $\mathrm{II_A}$), the susceptibility is exceptionally high.

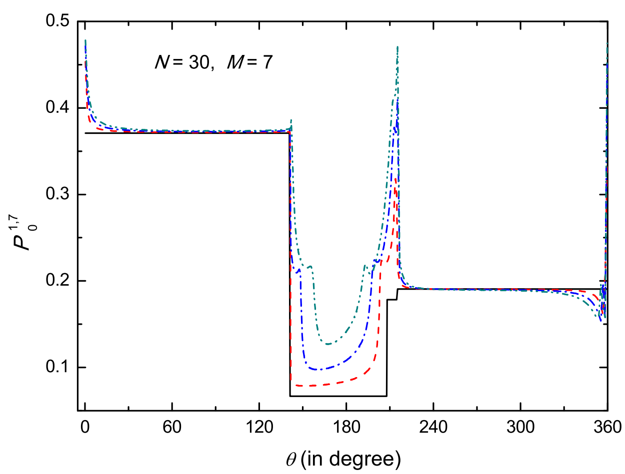

FIG. 5: (Color online.) $P_0^{1,7}$ against θ with $M = 7$ and $N = 30$. The solid, dash, dash-dot and dash-dot-dot curves have $q' = 0$, 0.02, 0.04, and 0.06, respectively.

When q' is given at four values the populations of $\mu = 0$ components against θ are shown in Fig. 5. Similar to Fig. 2, the pA- and pB- phases respond to the increase of q' more sensitively. The set of curves in $\mathrm{II_A}$ and $\mathrm{II_B}$

TABLE II: The four phases of the ground states with $M \neq 0$. Accordingly, the scope of θ is divided into four connected regions, the right border of each region is given in the last row. v_0 is the smallest even (odd) integer $\geq M/2$ if N is even (odd), and (v_g, F_g) are the quantum numbers of the ground states.

region	I (f-phase)	II$_\mathrm{A}$ (pA-phase)	II$_\mathrm{B}$ (pB-phase)	III (c-phase)
(v_g, F_g)	$(N, 2N)$	$(v_0, 2v_0)$	(v_0, M)	(N, M)
right border	$\arctan\{\frac{-7[N(N+3)-v_0(v_0+3)]}{10[N(N-2)-v_0(v_0-2)]}\} < \pi$	π	$\arctan(\frac{7}{10}) > \pi$	2π

will become even higher when N is larger. In particular, when θ is close to 0 (2π), and close to the narrow domain II$_\mathrm{B}$, the sensitivity is very high. Obviously, the sensitivity emerging in Fig. 5 is closely related to the susceptibility shown in Fig. 4.

It was found that the number of nonzero non-diagonal matrix elements $Q^M_{\bar{v}_g\bar{F}_g,v_gF_g}$ would be in general more than one when $M \neq 0$. Therefore, there are more than one elementary modes of excitation (i.e., Eq. (6) will contain more than two terms). The actual mode is a mixing of them. It was found that, when q' is very small, the ground state $\psi^{q'}_{1,M}$ in region I (f-phase), similar to Eq. (6), is essentially $\Psi_{N,2N,M} + b\Psi_{N,2N-2,M}$ where b is a small constant. For a numerical example, when $N = 30$, $M = 7$, $q' = 0.005$ and $\theta = \pi/2$, $b = -0.0019$. This value will become larger when θ is close to zero (say, $b = 0.1072$ if $\theta = \pi/180$). In general, $|b|$ will decrease a little when N becomes larger. In the region II$_\mathrm{A}$ (pA-phase) and for $M = 7$ as an example, $\psi^{q'}_{1,7} \approx \Psi_{4,8,7} + \sum_F b_F\Psi_{6,F,7}$, where F is ranged from 7 to 10 and b_F are in the order of 1/10 or smaller. When θ is close to the left border of II$_\mathrm{A}$, only $|b_{10}|$ is relatively larger (meanwhile $\sin\theta$ is positive and the components with a smaller F will have a higher energy and therefore can be neglected, refer to Eq. (3)). On the contrary, when θ is close to the right border, only $|b_7|$ is relatively larger (meanwhile $\sin\theta$ is negative). In II$_\mathrm{B}$ (pB-phase) $\psi^{q'}_{1,7} \approx \Psi_{6,7,7} + b_\alpha\Psi_{6,8,7} + b_\beta\Psi_{8,7,7}$. In III ($c$-phase) $\psi^{q'}_{1,7} \approx \Psi_{N,7,7} + b_\gamma\Psi_{N-2,7,7} + b_\delta\Psi_{N,8,7}$. The related coefficients in the two expansion are all small. When θ is close to the left border of III, the gap associated with the change of v from N to $N-2$ is very small, and therefore $|b_\gamma| >> |b_\delta|$. On the contrary, when θ is close to the right border 2π, the gap associated with the change of F from 7 to 8 is very small, therefore $|b_\gamma| << |b_\delta|$.

When q' is large, it was found that $\lim_{q'\to\infty}\langle 0, M, N-M, 0, 0|\psi^{q'}_{1M}\rangle^2 = 1$ disregarding how θ is, i.e., all the phases tend to the same Fock-state $|0, M, N-M, 0, 0\rangle$. Accordingly, $P^{1,M}_\mu \to M/N$ (if $\mu = 1$), $(N-M)/N$ (if $\mu = 0$), and 0 (otherwise). As examples, $P^{1,7}_1$and $P^{1,4}_1$ against q' are shown in Fig. 6 and Fig. 7, respectively. In both figures the solid curves (f-phase) are inert to q', the dash-dot curves (c-phase) respond to q' mildly. However, the dash curves (pA-phase in Fig. 6 and p-phase in Fig. 7) are more sensitive to q'. In particular, at some critical points, abrupt changes are found in the dash curves implying a sudden transition in spin-structure (this abrupt change is not found in the previous cases of $M = 0$). In order to understand the transition better, the associated ground state is expanded as $\psi^{q'}_{1,M} = \sum_{v,F} b_{vF}\Psi_{vFM}$. Then, we define $W_v = \sum_F (b_{vF})^2$ which is the weight of having v unpaired particles. For the dash curve in Fig. 6, we found that $W_4 = 1$ at $q' = 0$. When q' in-

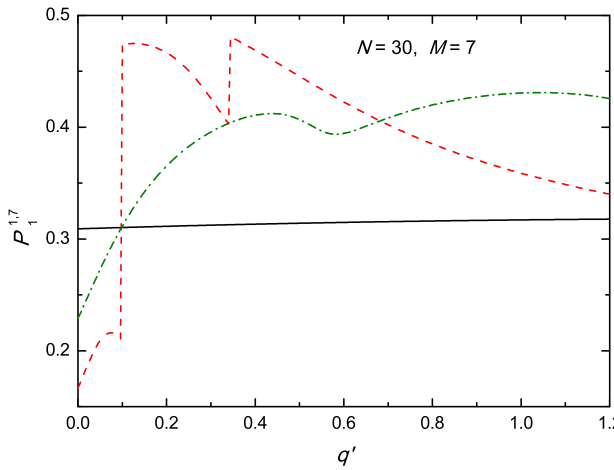

FIG. 6: (Color online.) $P^{1,7}_1$ against q' with $M = 7$ and $N = 30$. The solid, dash, and dash-dot curves have $\theta = \pi/3$, π, and $5\pi/3$ to represent the f-, pA-, and c-phases, respectively.

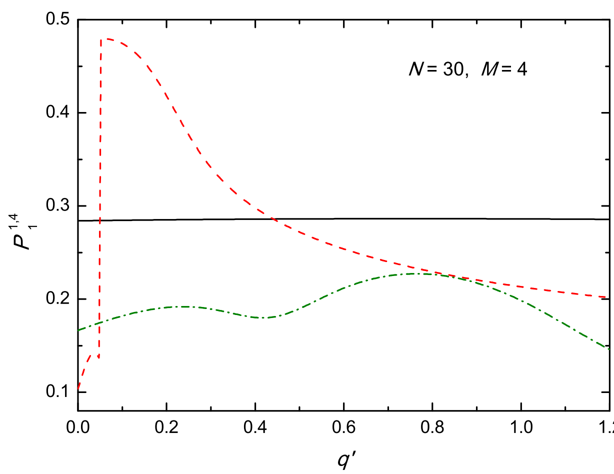

FIG. 7: (Color online.) $P^{1,4}_1$ against q' with $M = 4$ and $N = 30$. The implications of the curves are the same as in Fig. 6. They are chosen to represent the f-, p-, and c-phases.

creases, W_4 decreases and W_6 increases gradually. When $q' \to q'_{C_1} = 0.098$ (the critical point of the first transition), $W_4 \approx 0.55$ and $W_6 \approx 0.39$. However, when q' crosses the critical point, W_4 sudden becomes nearly zero while W_6 jumps up to 0.90. Thus the transition is characterized by a great decrease of W_4 together with a great increase of W_6. In Fig. 7, the dash curve have $W_2 = 1$ at $q' = 0$. The first and unique critical point $q'_{C_1} = 0.050$. Similarly, the associated transition is characterized by a great decrease of W_2 together with a great increase of W_4. When q' is larger than the range of Figs. 6 and 7, no further critical points are found. Instead, all the curves tend smoothly to their limit M/N.

VI. FINAL REMARKS

The effect of a magnetic field on the spin-structures of the ground states of small spin-2 condensates is studied. The elementary modes of excitation caused by the quadratic Zeeman term are found. The fidelity susceptibility and the populations of spin-components are calculated. Mostly the case $N = 30$ is studied in detail. The effect caused by increasing and decreasing N is also discussed. The following points are mentioned.

(i) The set of eigenstates Ψ_{vFM} with the $U(5) \supset SO(5) \supset SO(3)$ symmetry are introduced to help the analysis. When the magnetic field is zero ($q' = 0$) and $M = 0$, the pair of quantum numbers of the ground states (v_g, F_g) have three choices associated with the three phases (f-, p-, and c-phases).[5] It turns out that, for each phase, among all the non-diagonal matrix elements of the quadratic Zeeman terms, only the one $Q^0_{\bar{v}_g \bar{F}_g, v_g F_g}$ is nonzero. It implies each phase has its unique mode of elementary excitation. Thus the response of the system to a weak field is clear. In particular, the variation of the fidelity susceptibility against θ can be understood in an analytical way.

(ii) For the case $M \neq 0$, the ground state in general has four phases, namely, the f-, pA-, pB-, and c-phases, each has in general more than one elementary excitation modes. However, when θ locates at some particular domains, one elementary mode will be dominant.

(iii) The calculation of the fidelity susceptibility against θ confirms the existence of the phases. The p-, pA- and pB-phases have in general a much higher susceptibility. In particular, when θ is close to the borders separating the phases, the susceptibility may be very high.

(iv) In addition to the fidelity susceptibility, the variation of the populations against the field has also been calculated. Both quantities provide similar message. In particular, the high sensitivity in the neighborhood of the borders is found again in the population. Since it has been suggested that a number of realistic species would have their parameters of interaction close to the borders, [2] since the population is easier to measure, the high sensitivity is useful to the accurate determination of the parameters of interactions.

(v) when $q' \to \infty$, all the ground states tend to the limit $|0, M, N-M, 0, 0\rangle$ disregarding how θ is. The process of tending to the limit is in general smooth. However, when $M \neq 0$ and q' is not very large (refer to Fig. 6), abrupt changes against q' are found. It implies that the field will cause not only a smooth variation of structure but also a phase transition at some critical points.

Acknowledgments

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[1] D. M. Stamper-Kurn, M. R. Andrews, A. P. Chikkatur, S. Inouye, H. -J. Miesner, J. Stenger, and W. Ketterle, Phys. Rev. Lett. **80**, 2027 (1998).
[2] C. V. Ciobanu, S. -K. Yip, and T. L. Ho, Phys. Rev. A **61**, 033607 (2000).
[3] M. Koashi and M. Ueda, Phys. Rev. Lett. **84**, 1066 (2000).
[4] M.Ueda and M.Koashi, Phys. Rev. A **65**, 063602 (2002).
[5] P. V. Isacker and S. Heinze, J. Phys. A: Math. Theor. **40**, 14811 (2007).
[6] N. Gemelke, E. Sarajlic, and S. Chu, preprint arXiv:1007.2677v1 [Cond-mat.quant-gas] (2010).
[7] Y. Z. He and C. G.Bao, Phys. Rev. A **83**, 033622 (2011).
[8] C. G. Bao, under consideration in Phys. Rev. A.
[9] H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).
[10] W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007).
[11] P. Zanardi, P. Giorda, and M. Cozzini, Phys. Rev. Lett. **99**, 100603 (2007).
[12] J. Stenger, S. Inouye, D. M. Stamper-Kurn, H. -J. Miesner, A. P. Chikkatur, and W.Ketterle, Nature (London) **396**, 345 (1998).
[13] A. Gheorghe and A. A. Raduta, J. Phys. A: Math. Gen. **37**, 10951 (2004).
[14] E. Chacón, M. Moshinsky, and R. T. Sharp, J. Math. Phys. **17**, 668 (1976).
[15] Some essential points of symmetry constraint on the pair (v, F) are the following. (i) $F > 2v$ is not allowed. (ii) When N is even, v must be even. If v is a multiple of 6 (including $v = 0$), $F = 1, 2$, and 5 are not allowed. If v is not, $F = 0, 1$, and 3 are not allowed. (iii) When N is odd, v must be odd. If $v - 3$ is a multiple of 6, $F = 1, 2$, and 5 are not allowed. If $v - 3$ is not, $F = 0, 1$, and 3 are not allowed. (iv) $(v, 2v - 1)$ is not allowed [13].
[16] In general, the form of the eigenstate Ψ_{vFM} is complicated. However, for the p-phase, Ψ_{000} is exactly proportional to $\tilde{S}[(\eta\eta)_0]^{N/2}$, where an even N is assumed,

$\tilde{S}$ is the operator of symmetrization, and the spins are two-by-two coupled to zero. Ψ_{220} is exactly proportional to $\tilde{S}(\eta\eta)_{20}[(\eta\eta)_0]^{N/2-1}$. Since they have a rather simple form, they can be easily expanded in terms of the Fock-states. Thereby the matrix element for the elementary excitation, $Q^0_{22,00}$, can have an analytical from. So, we can obtain numerical result even when N is larger (say, $N = 500$), and we find that $Q^0_{22,00}$ is roughly proportional to N.

[17] In fact, when θ is close to θ_{pc}, not only the gap decreases, a group of states having different v but the same $F = F_g$ fall down and the ground state becomes to be nearly degenerate. This is also an important factor to raise the susceptibility.